

# Intrinsic square functions on the weighted Morrey spaces

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## Abstract

In this paper, we will study the boundedness properties of intrinsic square functions including the Lusin area integral, Littlewood-Paley  $g$ -function and  $g_\lambda^*$ -function on the weighted Morrey spaces  $L^{p,\kappa}(w)$  for  $1 < p < \infty$  and  $0 < \kappa < 1$ . The corresponding commutators generated by  $BMO(\mathbb{R}^n)$  functions and intrinsic square functions are also discussed.

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## 1 Introduction and main results

Let  $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0, \infty)$  and  $\varphi_t(x) = t^{-n}\varphi(x/t)$ . The classical square function (Lusin area integral) is a familiar object. If  $u(x, t) = P_t * f(x)$  is the Poisson integral of  $f$ , where  $P_t(x) = c_n \frac{t}{(t^2 + |x|^2)^{(n+1)/2}}$  denotes the Poisson kernel in  $\mathbb{R}_+^{n+1}$ . Then we define the classical square function (Lusin area integral)  $S(f)$  by

$$S(f)(x) = \left( \iint_{\Gamma(x)} |\nabla u(y, t)|^2 t^{1-n} dy dt \right)^{1/2},$$

where  $\Gamma(x)$  denotes the usual cone of aperture one:

$$\Gamma(x) = \{(y, t) \in \mathbb{R}_+^{n+1} : |x - y| < t\}$$

and

$$|\nabla u(y, t)| = \left| \frac{\partial u}{\partial t} \right|^2 + \sum_{j=1}^n \left| \frac{\partial u}{\partial y_j} \right|^2.$$

We can similarly define a cone of aperture  $\beta$  for any  $\beta > 0$ :

$$\Gamma_\beta(x) = \{(y, t) \in \mathbb{R}_+^{n+1} : |x - y| < \beta t\},$$

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and corresponding square function

$$S_\beta(f)(x) = \left( \iint_{\Gamma_\beta(x)} |\nabla u(y, t)|^2 t^{1-n} dy dt \right)^{1/2}.$$

The Littlewood-Paley  $g$ -function (could be viewed as a “zero-aperture” version of  $S(f)$ ) and the  $g_\lambda^*$ -function (could be viewed as an “infinite aperture” version of  $S(f)$ ) are defined respectively by

$$g(f)(x) = \left( \int_0^\infty |\nabla u(x, t)|^2 t dt \right)^{1/2}$$

and

$$g_\lambda^*(f)(x) = \left( \iint_{\mathbb{R}_+^{n+1}} \left( \frac{t}{t + |x - y|} \right)^{\lambda n} |\nabla u(y, t)|^2 t^{1-n} dy dt \right)^{1/2}.$$

The modern (real-variable) variant of  $S_\beta(f)$  can be defined in the following way. Let  $\psi \in C^\infty(\mathbb{R}^n)$  be real, radial, have support contained in  $\{x : |x| \leq 1\}$ , and  $\int_{\mathbb{R}^n} \psi(x) dx = 0$ . The continuous square function  $S_{\psi, \beta}(f)$  is defined by

$$S_{\psi, \beta}(f)(x) = \left( \iint_{\Gamma_\beta(x)} |f * \psi_t(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}.$$

In 2007, Wilson [24] introduced a new square function called intrinsic square function which is universal in a sense (see also [25]). This function is independent of any particular kernel  $\psi$ , and it dominates pointwise all the above defined square functions. On the other hand, it is not essentially larger than any particular  $S_{\psi, \beta}(f)$ . For  $0 < \alpha \leq 1$ , let  $\mathcal{C}_\alpha$  be the family of functions  $\varphi$  defined on  $\mathbb{R}^n$  such that  $\varphi$  has support containing in  $\{x \in \mathbb{R}^n : |x| \leq 1\}$ ,  $\int_{\mathbb{R}^n} \varphi(x) dx = 0$ , and for all  $x, x' \in \mathbb{R}^n$ ,

$$|\varphi(x) - \varphi(x')| \leq |x - x'|^\alpha.$$

For  $(y, t) \in \mathbb{R}_+^{n+1}$  and  $f \in L_{loc}^1(\mathbb{R}^n)$ , we set

$$A_\alpha(f)(y, t) = \sup_{\varphi \in \mathcal{C}_\alpha} |f * \varphi_t(y)| = \sup_{\varphi \in \mathcal{C}_\alpha} \left| \int_{\mathbb{R}^n} \varphi_t(y - z) f(z) dz \right|.$$

Then we define the intrinsic square function of  $f$  (of order  $\alpha$ ) by the formula

$$\mathcal{S}_\alpha(f)(x) = \left( \iint_{\Gamma(x)} \left( A_\alpha(f)(y, t) \right)^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}.$$

We can also define varying-aperture versions of  $\mathcal{S}_\alpha(f)$  by the formula

$$\mathcal{S}_{\alpha, \beta}(f)(x) = \left( \iint_{\Gamma_\beta(x)} \left( A_\alpha(f)(y, t) \right)^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}.$$

The intrinsic Littlewood-Paley  $g$ -function and the intrinsic  $g_\lambda^*$ -function will be defined respectively by

$$g_\alpha(f)(x) = \left( \int_0^\infty \left( A_\alpha(f)(x, t) \right)^2 \frac{dt}{t} \right)^{1/2}$$

and

$$g_{\lambda, \alpha}^*(f)(x) = \left( \iint_{\mathbb{R}_+^{n+1}} \left( \frac{t}{t + |x - y|} \right)^{\lambda n} \left( A_\alpha(f)(y, t) \right)^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}.$$

In [25], Wilson proved the following result.

**Theorem A.** *Let  $0 < \alpha \leq 1$ ,  $1 < p < \infty$  and  $w \in A_p$  (Muckenhoupt weight class). Then there exists a constant  $C > 0$  independent of  $f$  such that*

$$\|\mathcal{S}_\alpha(f)\|_{L_w^p} \leq C \|f\|_{L_w^p}.$$

Moreover, in [14], Lerner showed sharp  $L_w^p$  norm inequalities for the intrinsic square functions in terms of the  $A_p$  characteristic constant of  $w$  for all  $1 < p < \infty$ . As for the boundedness of intrinsic square functions on the weighted Hardy spaces  $H_w^p(\mathbb{R}^n)$  for  $n/(n + \alpha) \leq p \leq 1$ , we refer the readers to [11], [22] and [23].

Let  $b$  be a locally integrable function on  $\mathbb{R}^n$ , in this paper, we will also consider the commutators generated by  $b$  and intrinsic square functions, which are defined respectively by the following expressions

$$[b, \mathcal{S}_\alpha](f)(x) = \left( \iint_{\Gamma(x)} \sup_{\varphi \in \mathcal{C}_\alpha} \left| \int_{\mathbb{R}^n} [b(x) - b(z)] \varphi_t(y - z) f(z) dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2},$$

$$[b, g_\alpha](f)(x) = \left( \int_0^\infty \sup_{\varphi \in \mathcal{C}_\alpha} \left| \int_{\mathbb{R}^n} [b(x) - b(y)] \varphi_t(x - y) f(y) dy \right|^2 \frac{dt}{t} \right)^{1/2},$$

and

$$[b, g_{\lambda, \alpha}^*](f)(x) = \left( \iint_{\mathbb{R}_+^{n+1}} \left( \frac{t}{t + |x - y|} \right)^{\lambda n} \sup_{\varphi \in \mathcal{C}_\alpha} \left| \int_{\mathbb{R}^n} [b(x) - b(z)] \varphi_t(y - z) f(z) dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}.$$

The classical Morrey spaces  $\mathcal{L}^{p, \lambda}$  were first introduced by Morrey in [15] to study the local behavior of solutions to second order elliptic partial differential equations. For the boundedness of the Hardy-Littlewood maximal operator, the fractional integral operator and the Calderón-Zygmund singular integral operator on these spaces, we refer the readers to [1, 3, 17]. For the properties and applications of classical Morrey spaces, see [6, 7, 8] and the references therein.

In 2009, Komori and Shirai [13] first defined the weighted Morrey spaces  $L^{p, \kappa}(w)$  which could be viewed as an extension of weighted Lebesgue spaces,

and studied the boundedness of the above classical operators on these weighted spaces. Recently, in [19], [20] and [21], we have established the continuity properties of some other operators on the weighted Morrey spaces  $L^{p,\kappa}(w)$ .

The purpose of this paper is to discuss the boundedness properties of intrinsic square functions and their commutators on the weighted Morrey spaces  $L^{p,\kappa}(w)$  for all  $1 < p < \infty$  and  $0 < \kappa < 1$ . Our main results in the paper are formulated as follows.

**Theorem 1.1.** *Let  $0 < \alpha \leq 1$ ,  $1 < p < \infty$ ,  $0 < \kappa < 1$  and  $w \in A_p$ . Then there is a constant  $C > 0$  independent of  $f$  such that*

$$\|\mathcal{S}_\alpha(f)\|_{L^{p,\kappa}(w)} \leq C\|f\|_{L^{p,\kappa}(w)}.$$

**Theorem 1.2.** *Let  $0 < \alpha \leq 1$ ,  $1 < p < \infty$ ,  $0 < \kappa < 1$  and  $w \in A_p$ . Suppose that  $b \in BMO(\mathbb{R}^n)$ , then there is a constant  $C > 0$  independent of  $f$  such that*

$$\|[b, \mathcal{S}_\alpha](f)\|_{L^{p,\kappa}(w)} \leq C\|f\|_{L^{p,\kappa}(w)}.$$

**Theorem 1.3.** *Let  $0 < \alpha \leq 1$ ,  $1 < p < \infty$ ,  $0 < \kappa < 1$  and  $w \in A_p$ . If  $\lambda > \max\{p, 3\}$ , then there is a constant  $C > 0$  independent of  $f$  such that*

$$\|g_{\lambda,\alpha}^*(f)\|_{L^{p,\kappa}(w)} \leq C\|f\|_{L^{p,\kappa}(w)}.$$

**Theorem 1.4.** *Let  $0 < \alpha \leq 1$ ,  $1 < p < \infty$ ,  $0 < \kappa < 1$  and  $w \in A_p$ . If  $b \in BMO(\mathbb{R}^n)$  and  $\lambda > \max\{p, 3\}$ , then there is a constant  $C > 0$  independent of  $f$  such that*

$$\|[b, g_{\lambda,\alpha}^*](f)\|_{L^{p,\kappa}(w)} \leq C\|f\|_{L^{p,\kappa}(w)}.$$

In [24], Wilson also showed that for any  $0 < \alpha \leq 1$ , the functions  $\mathcal{S}_\alpha(f)(x)$  and  $g_\alpha(f)(x)$  are pointwise comparable. Thus, as a direct consequence of Theorems 1.1 and 1.2, we obtain the following

**Corollary 1.5.** *Let  $0 < \alpha \leq 1$ ,  $1 < p < \infty$ ,  $0 < \kappa < 1$  and  $w \in A_p$ . Then there is a constant  $C > 0$  independent of  $f$  such that*

$$\|g_\alpha(f)\|_{L^{p,\kappa}(w)} \leq C\|f\|_{L^{p,\kappa}(w)}.$$

**Corollary 1.6.** *Let  $0 < \alpha \leq 1$ ,  $1 < p < \infty$ ,  $0 < \kappa < 1$  and  $w \in A_p$ . Suppose that  $b \in BMO(\mathbb{R}^n)$ , then there is a constant  $C > 0$  independent of  $f$  such that*

$$\|[b, g_\alpha](f)\|_{L^{p,\kappa}(w)} \leq C\|f\|_{L^{p,\kappa}(w)}.$$

## 2 Notations and definitions

The classical  $A_p$  weight theory was first introduced by Muckenhoupt in the study of weighted  $L^p$  boundedness of Hardy-Littlewood maximal functions in [16]. A weight  $w$  is a nonnegative, locally integrable function on  $\mathbb{R}^n$ ,  $B = B(x_0, r_B)$

denotes the ball with the center  $x_0$  and radius  $r_B$ . Given a ball  $B$  and  $\lambda > 0$ ,  $\lambda B$  denotes the ball with the same center as  $B$  whose radius is  $\lambda$  times that of  $B$ . For a given weight function  $w$  and a measurable set  $E$ , we also denote the Lebesgue measure of  $E$  by  $|E|$  and the weighted measure of  $E$  by  $w(E)$ , where  $w(E) = \int_E w(x) dx$ . We say that  $w$  is in the Muckenhoupt class  $A_p$  with  $1 < p < \infty$ , if there exists a constant  $C > 0$  such that for every ball  $B \subseteq \mathbb{R}^n$ ,

$$\left( \frac{1}{|B|} \int_B w(x) dx \right) \left( \frac{1}{|B|} \int_B w(x)^{-1/(p-1)} dx \right)^{p-1} \leq C.$$

The smallest constant  $C$  such that the above inequality holds is called the  $A_p$  characteristic constant of  $w$  and denoted by  $[w]_{A_p}$ . A weight function  $w$  is said to belong to the reverse Hölder class  $RH_r$  if there exist two constants  $r > 1$  and  $C > 0$  such that the following reverse Hölder inequality holds for every ball  $B \subseteq \mathbb{R}^n$ .

$$\left( \frac{1}{|B|} \int_B w(x)^r dx \right)^{1/r} \leq C \left( \frac{1}{|B|} \int_B w(x) dx \right).$$

We state the following results that we will use frequently in the sequel.

**Lemma 2.1** ([9]). *Let  $w \in A_p$  with  $1 < p < \infty$ . Then, for any ball  $B$ , there exists an absolute constant  $C > 0$  such that*

$$w(2B) \leq C w(B).$$

*In general, for any  $\lambda > 1$ , we have*

$$w(\lambda B) \leq C \cdot \lambda^{np} w(B),$$

*where  $C$  does not depend on  $B$  nor on  $\lambda$ .*

**Lemma 2.2** ([10]). *Let  $w \in RH_r$  with  $r > 1$ . Then there exists a constant  $C > 0$  such that*

$$\frac{w(E)}{w(B)} \leq C \left( \frac{|E|}{|B|} \right)^{(r-1)/r}$$

*for any measurable subset  $E$  of a ball  $B$ .*

Given a weight function  $w$  on  $\mathbb{R}^n$ , for  $1 < p < \infty$ , we denote by  $L_w^p(\mathbb{R}^n)$  the space of all functions satisfying

$$\|f\|_{L_w^p} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty.$$

A locally integrable function  $b$  is said to be in  $BMO(\mathbb{R}^n)$  if

$$\|b\|_* = \sup_B \frac{1}{|B|} \int_B |b(x) - b_B| dx < \infty,$$

where  $b_B$  stands for the average of  $b$  on  $B$ , i.e.,  $b_B = \frac{1}{|B|} \int_B b(y) dy$  and the supremum is taken over all balls  $B$  in  $\mathbb{R}^n$ .

**Theorem 2.3** ([5, 12]). Assume that  $b \in BMO(\mathbb{R}^n)$ . Then for any  $1 \leq p < \infty$ , we have

$$\sup_B \left( \frac{1}{|B|} \int_B |b(x) - b_B|^p dx \right)^{1/p} \leq C \|b\|_*.$$

**Definition 2.4** ([13]). Let  $1 \leq p < \infty$ ,  $0 < \kappa < 1$  and  $w$  be a weight function. Then the weighted Morrey space is defined by

$$L^{p,\kappa}(w) = \{f \in L^p_{loc}(w) : \|f\|_{L^{p,\kappa}(w)} < \infty\},$$

where

$$\|f\|_{L^{p,\kappa}(w)} = \sup_B \left( \frac{1}{w(B)^\kappa} \int_B |f(x)|^p w(x) dx \right)^{1/p}$$

and the supremum is taken over all balls  $B$  in  $\mathbb{R}^n$ .

Throughout this article, we will use  $C$  to denote a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence. Moreover, we will denote the conjugate exponent of  $p > 1$  by  $p' = p/(p-1)$ .

### 3 Proofs of Theorems 1.1 and 1.2

*Proof of Theorem 1.1.* Fix a ball  $B = B(x_0, r_B) \subseteq \mathbb{R}^n$  and decompose  $f = f_1 + f_2$ , where  $f_1 = f \chi_{2B}$ ,  $\chi_{2B}$  denotes the characteristic function of  $2B$ . Since  $\mathcal{S}_\alpha(0 < \alpha \leq 1)$  is a sublinear operator, then we have

$$\begin{aligned} & \frac{1}{w(B)^{\kappa/p}} \left( \int_B |\mathcal{S}_\alpha(f)(x)|^p w(x) dx \right)^{1/p} \\ & \leq \frac{1}{w(B)^{\kappa/p}} \left( \int_B |\mathcal{S}_\alpha(f_1)(x)|^p w(x) dx \right)^{1/p} + \frac{1}{w(B)^{\kappa/p}} \left( \int_B |\mathcal{S}_\alpha(f_2)(x)|^p w(x) dx \right)^{1/p} \\ & = I_1 + I_2. \end{aligned}$$

Theorem A and Lemma 2.1 imply

$$\begin{aligned} I_1 & \leq C \cdot \frac{1}{w(B)^{\kappa/p}} \left( \int_{2B} |f(x)|^p w(x) dx \right)^{1/p} \\ & \leq C \|f\|_{L^{p,\kappa}(w)} \cdot \frac{w(2B)^{\kappa/p}}{w(B)^{\kappa/p}} \\ & \leq C \|f\|_{L^{p,\kappa}(w)}. \end{aligned}$$

We now turn to estimate the other term  $I_2$ . For any  $\varphi \in \mathcal{C}_\alpha$ ,  $0 < \alpha \leq 1$  and  $(y, t) \in \Gamma(x)$ , we have

$$\begin{aligned} |f_2 * \varphi_t(y)| &= \left| \int_{(2B)^c} \varphi_t(y-z) f(z) dz \right| \\ &\leq C \cdot t^{-n} \int_{(2B)^c \cap \{z: |y-z| \leq t\}} |f(z)| dz \\ &\leq C \cdot t^{-n} \sum_{k=1}^{\infty} \int_{(2^{k+1}B \setminus 2^k B) \cap \{z: |y-z| \leq t\}} |f(z)| dz. \end{aligned} \quad (1)$$

For any  $x \in B$ ,  $(y, t) \in \Gamma(x)$  and  $z \in (2^{k+1}B \setminus 2^k B) \cap B(y, t)$ , then by a direct computation, we can easily see that

$$2t \geq |x - y| + |y - z| \geq |x - z| \geq |z - x_0| - |x - x_0| \geq 2^{k-1}r_B.$$

Thus, by using the above inequality (1) and Minkowski's integral inequality, we deduce

$$\begin{aligned} |\mathcal{S}_\alpha(f_2)(x)| &= \left( \iint_{\Gamma(x)} \sup_{\varphi \in \mathcal{C}_\alpha} |f_2 * \varphi_t(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} \\ &\leq C \left( \int_{2^{k-2}r_B}^{\infty} \int_{|x-y| < t} \left| t^{-n} \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |f(z)| dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} \\ &\leq C \left( \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |f(z)| dz \right) \left( \int_{2^{k-2}r_B}^{\infty} \frac{dt}{t^{2n+1}} \right)^{1/2} \\ &\leq C \sum_{k=1}^{\infty} \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B \setminus 2^k B} |f(z)| dz. \end{aligned}$$

It follows from Hölder's inequality and the  $A_p$  condition that

$$\begin{aligned} \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} |f(z)| dz &\leq \frac{1}{|2^{k+1}B|} \left( \int_{2^{k+1}B} |f(z)|^p w(z) dz \right)^{1/p} \left( \int_{2^{k+1}B} w(z)^{-p'/p} dz \right)^{1/p'} \\ &\leq C \|f\|_{L^{p,\kappa}(w)} \cdot w(2^{k+1}B)^{(\kappa-1)/p}. \end{aligned} \quad (2)$$

Hence

$$I_2 \leq C \|f\|_{L^{p,\kappa}(w)} \sum_{k=1}^{\infty} \frac{w(B)^{(1-\kappa)/p}}{w(2^{k+1}B)^{(1-\kappa)/p}}.$$

Since  $w \in A_p$  with  $1 < p < \infty$ , then there exists a number  $r > 1$  such that  $w \in RH_r$ . Consequently, by using Lemma 2.2, we can get

$$\frac{w(B)}{w(2^{k+1}B)} \leq C \left( \frac{|B|}{|2^{k+1}B|} \right)^{(r-1)/r}. \quad (3)$$

Therefore

$$\begin{aligned} I_2 &\leq C \|f\|_{L^{p,\kappa}(w)} \sum_{k=1}^{\infty} \left( \frac{1}{2^{kn}} \right)^{(1-\kappa)(r-1)/pr} \\ &\leq C \|f\|_{L^{p,\kappa}(w)}, \end{aligned}$$

where the last series is convergent since  $(1-\kappa)(r-1)/pr > 0$ . Combining the above estimates for  $I_1$  and  $I_2$  and taking the supremum over all balls  $B \subseteq \mathbb{R}^n$ , we complete the proof of Theorem 1.1.  $\square$

Given a real-valued function  $b \in BMO(\mathbb{R}^n)$ , we shall follow the idea developed in [2, 4] and denote  $F(\xi) = e^{\xi[b(x)-b(z)]}$ ,  $\xi \in \mathbb{C}$ . Then by the analyticity of  $F(\xi)$  on  $\mathbb{C}$  and the Cauchy integral formula, we get

$$\begin{aligned} b(x) - b(z) &= F'(0) = \frac{1}{2\pi i} \int_{|\xi|=1} \frac{F(\xi)}{\xi^2} d\xi \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{e^{i\theta}[b(x)-b(z)]} e^{-i\theta} d\theta. \end{aligned}$$

Thus, for any  $\varphi \in \mathcal{C}_\alpha$ ,  $0 < \alpha \leq 1$ , we obtain

$$\begin{aligned} \left| \int_{\mathbb{R}^n} [b(x) - b(z)] \varphi_t(y-z) f(z) dz \right| &= \left| \frac{1}{2\pi} \int_0^{2\pi} \left( \int_{\mathbb{R}^n} \varphi_t(y-z) e^{-e^{i\theta}b(z)} f(z) dz \right) e^{e^{i\theta}b(x)} e^{-i\theta} d\theta \right| \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \sup_{\varphi \in \mathcal{C}_\alpha} \left| \int_{\mathbb{R}^n} \varphi_t(y-z) e^{-e^{i\theta}b(z)} f(z) dz \right| e^{\cos \theta \cdot b(x)} d\theta \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} A_\alpha(e^{-e^{i\theta}b} \cdot f)(y, t) \cdot e^{\cos \theta \cdot b(x)} d\theta. \end{aligned}$$

So we have

$$\begin{aligned} |[b, \mathcal{S}_\alpha](f)(x)| &\leq \frac{1}{2\pi} \int_0^{2\pi} \mathcal{S}_\alpha(e^{-e^{i\theta}b} \cdot f)(x) \cdot e^{\cos \theta \cdot b(x)} d\theta, \\ |[b, g_{\lambda,\alpha}^*](f)(x)| &\leq \frac{1}{2\pi} \int_0^{2\pi} g_{\lambda,\alpha}^*(e^{-e^{i\theta}b} \cdot f)(x) \cdot e^{\cos \theta \cdot b(x)} d\theta. \end{aligned}$$

Then, by using the same arguments as in [4], we can also show the following

**Theorem 3.1.** *Let  $0 < \alpha \leq 1$ ,  $1 < p < \infty$  and  $w \in A_p$ . Then the commutators  $[b, \mathcal{S}_\alpha]$  and  $[b, g_{\lambda,\alpha}^*]$  are all bounded from  $L_w^p(\mathbb{R}^n)$  into itself whenever  $b \in BMO(\mathbb{R}^n)$ .*

*Proof of Theorem 1.2.* Fix a ball  $B = B(x_0, r_B) \subseteq \mathbb{R}^n$ . Let  $f = f_1 + f_2$ , where  $f_1 = f\chi_{2B}$ . Then we can write

$$\begin{aligned} &\frac{1}{w(B)^{\kappa/p}} \left( \int_B |[b, \mathcal{S}_\alpha](f)(x)|^p w(x) dx \right)^{1/p} \\ &\leq \frac{1}{w(B)^{\kappa/p}} \left( \int_B |[b, \mathcal{S}_\alpha](f_1)(x)|^p w(x) dx \right)^{1/p} + \frac{1}{w(B)^{\kappa/p}} \left( \int_B |[b, \mathcal{S}_\alpha](f_2)(x)|^p w(x) dx \right)^{1/p} \\ &= J_1 + J_2. \end{aligned}$$



Applying Theorem 3.1 and Lemma 2.1, we thus obtain

$$\begin{aligned}
J_1 &\leq C \cdot \frac{1}{w(B)^{\kappa/p}} \left( \int_{2B} |f(x)|^p w(x) dx \right)^{1/p} \\
&\leq C \|f\|_{L^{p,\kappa}(w)} \cdot \frac{w(2B)^{\kappa/p}}{w(B)^{\kappa/p}} \\
&\leq C \|f\|_{L^{p,\kappa}(w)}.
\end{aligned} \tag{4}$$

We now turn to deal with the term  $J_2$ . For any given  $x \in B$  and  $(y, t) \in \Gamma(x)$ , we have

$$\begin{aligned}
\sup_{\varphi \in \mathcal{C}_\alpha} \left| \int_{\mathbb{R}^n} [b(x) - b(z)] \varphi_t(y - z) f_2(z) dz \right| &\leq |b(x) - b_B| \cdot \sup_{\varphi \in \mathcal{C}_\alpha} \left| \int_{\mathbb{R}^n} \varphi_t(y - z) f_2(z) dz \right| \\
&\quad + \sup_{\varphi \in \mathcal{C}_\alpha} \left| \int_{\mathbb{R}^n} [b(z) - b_B] \varphi_t(y - z) f_2(z) dz \right|
\end{aligned}$$

Hence

$$\begin{aligned}
|[b, \mathcal{S}_\alpha](f_2)(x)| &\leq |b(x) - b_B| \cdot \mathcal{S}_\alpha(f_2)(x) \\
&\quad + \left( \iint_{\Gamma(x)} \sup_{\varphi \in \mathcal{C}_\alpha} \left| \int_{\mathbb{R}^n} [b(z) - b_B] \varphi_t(y - z) f_2(z) dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} \\
&= \text{I} + \text{II}.
\end{aligned}$$

In the proof of Theorem 1.1, we have already proved that for any  $x \in B$ ,

$$|\mathcal{S}_\alpha(f_2)(x)| \leq C \|f\|_{L^{p,\kappa}(w)} \cdot \sum_{k=1}^{\infty} w(2^{k+1}B)^{(\kappa-1)/p}.$$

Consequently

$$\begin{aligned}
&\frac{1}{w(B)^{\kappa/p}} \left( \int_B |f(x)|^p w(x) dx \right)^{1/p} \\
&\leq C \|f\|_{L^{p,\kappa}(w)} \frac{1}{w(B)^{\kappa/p}} \cdot \sum_{k=1}^{\infty} w(2^{k+1}B)^{(\kappa-1)/p} \cdot \left( \int_B |b(x) - b_B|^p w(x) dx \right)^{1/p} \\
&= C \|f\|_{L^{p,\kappa}(w)} \sum_{k=1}^{\infty} \frac{w(B)^{(1-\kappa)/p}}{w(2^{k+1}B)^{(1-\kappa)/p}} \cdot \left( \frac{1}{w(B)} \int_B |b(x) - b_B|^p w(x) dx \right)^{1/p}.
\end{aligned}$$

Using the same arguments as that of Theorem 1.1, we can see that the above summation is bounded by a constant. Hence

$$\frac{1}{w(B)^{\kappa/p}} \left( \int_B |f(x)|^p w(x) dx \right)^{1/p} \leq C \|f\|_{L^{p,\kappa}(w)} \left( \frac{1}{w(B)} \int_B |b(x) - b_B|^p w(x) dx \right)^{1/p}.$$

Since  $w \in A_p$ , as before, we know that there exists a number  $r > 1$  such that  $w \in RH_r$ . Thus by Hölder's inequality and Theorem 2.3, we deduce

$$\begin{aligned}
& \left( \frac{1}{w(B)} \int_B |b(x) - b_B|^p w(x) dx \right)^{1/p} \\
& \leq \frac{1}{w(B)^{1/p}} \left( \int_B |b(x) - b_B|^{pr'} dx \right)^{1/(pr')} \left( \int_B w(x)^r dx \right)^{1/(pr)} \\
& \leq C \cdot \left( \frac{1}{|B|} \int_B |b(x) - b_B|^{pr'} dx \right)^{1/(pr')} \\
& \leq C \|b\|_*.
\end{aligned} \tag{5}$$

So we have

$$\frac{1}{w(B)^{\kappa/p}} \left( \int_B \mathbf{I}^p w(x) dx \right)^{1/p} \leq C \|b\|_* \|f\|_{L^{p,\kappa}(w)}. \tag{6}$$

On the other hand

$$\begin{aligned}
\Pi &= \left( \iint_{\Gamma(x)} \sup_{\varphi \in \mathcal{C}_\alpha} \left| \int_{(2B)^c} [b(z) - b_B] \varphi_t(y-z) f(z) dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \\
&\leq C \left( \iint_{\Gamma(x)} \left| t^{-n} \sum_{k=1}^{\infty} \int_{(2^{k+1}B \setminus 2^k B) \cap \{z: |y-z| \leq t\}} |b(z) - b_B| |f(z)| dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \\
&\leq C \left( \iint_{\Gamma(x)} \left| t^{-n} \sum_{k=1}^{\infty} \int_{(2^{k+1}B \setminus 2^k B) \cap \{z: |y-z| \leq t\}} |b(z) - b_{2^{k+1}B}| |f(z)| dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \\
&+ C \left( \iint_{\Gamma(x)} \left| t^{-n} \sum_{k=1}^{\infty} |b_{2^{k+1}B} - b_B| \cdot \int_{(2^{k+1}B \setminus 2^k B) \cap \{z: |y-z| \leq t\}} |f(z)| dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \\
&= \text{III} + \text{IV}.
\end{aligned}$$

An application of Hölder's inequality gives us that

$$\begin{aligned}
& \int_{2^{k+1}B \setminus 2^k B} |b(z) - b_{2^{k+1}B}| |f(z)| dz \\
& \leq \left( \int_{2^{k+1}B} |b(z) - b_{2^{k+1}B}|^{p'} w(z)^{-p'/p} dz \right)^{1/p'} \left( \int_{2^{k+1}B} |f(z)|^p w(z) dz \right)^{1/p} \\
& \leq C \|f\|_{L^{p,\kappa}(w)} \cdot w(2^{k+1}B)^{\kappa/p} \left( \int_{2^{k+1}B} |b(z) - b_{2^{k+1}B}|^{p'} w(z)^{-p'/p} dz \right)^{1/p'}. \tag{7}
\end{aligned}$$

If we set  $\nu(z) = w(z)^{-p'/p} = w(z)^{1-p'}$ , then we have  $\nu \in A_{p'}$  because  $w \in A_p$  (see [9]). Following along the same lines as in the proof of (5), we can also show

$$\left( \frac{1}{\nu(2^{k+1}B)} \int_{2^{k+1}B} |b(z) - b_{2^{k+1}B}|^{p'} \nu(z) dz \right)^{1/p'} \leq C \|b\|_*. \tag{8}$$

Substituting the above inequality (8) into (7), we thus obtain

$$\begin{aligned} \int_{2^{k+1}B} |b(z) - b_{2^{k+1}B}| |f(z)| dz &\leq C \|b\|_* \|f\|_{L^{p,\kappa}(w)} \cdot w(2^{k+1}B)^{\kappa/p} \nu(2^{k+1}B)^{1/p'} \\ &\leq C \|b\|_* \|f\|_{L^{p,\kappa}(w)} \cdot |2^{k+1}B| w(2^{k+1}B)^{(\kappa-1)/p}. \end{aligned}$$

In addition, we note that in this case,  $t \geq 2^{k-2}r_B$  as in Theorem 1.1. From the above inequality, it follows that

$$\begin{aligned} \text{III} &\leq C \left( \int_{2^{k-2}r_B}^{\infty} \int_{|x-y|<t} \left| t^{-n} \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |b(z) - b_{2^{k+1}B}| |f(z)| dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} \\ &\leq C \left( \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |b(z) - b_{2^{k+1}B}| |f(z)| dz \right) \left( \int_{2^{k-2}r_B}^{\infty} \frac{dt}{t^{2n+1}} \right)^{1/2} \\ &\leq C \|b\|_* \|f\|_{L^{p,\kappa}(w)} \cdot w(2^{k+1}B)^{(\kappa-1)/p}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{w(B)^{\kappa/p}} \left( \int_B \text{III}^p w(x) dx \right)^{1/p} &\leq C \|b\|_* \|f\|_{L^{p,\kappa}(w)} \sum_{k=1}^{\infty} \frac{w(B)^{(1-\kappa)/p}}{w(2^{k+1}B)^{(1-\kappa)/p}} \\ &\leq C \|b\|_* \|f\|_{L^{p,\kappa}(w)}. \end{aligned} \quad (9)$$

Now let us deal with the last term IV. Since  $b \in BMO(\mathbb{R}^n)$ , then a simple calculation shows that

$$|b_{2^{k+1}B} - b_B| \leq C \cdot (k+1) \|b\|_*. \quad (10)$$

It follows from the inequalities (2) and (10) that

$$\begin{aligned} \text{IV} &\leq C \left( \int_{2^{k-2}r_B}^{\infty} \int_{|x-y|<t} \left| t^{-n} \sum_{k=1}^{\infty} |b_{2^{k+1}B} - b_B| \cdot \int_{2^{k+1}B \setminus 2^k B} |f(z)| dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} \\ &\leq C \|b\|_* \left( \sum_{k=1}^{\infty} (k+1) \cdot \int_{2^{k+1}B \setminus 2^k B} |f(z)| dz \right) \left( \int_{2^{k-2}r_B}^{\infty} \frac{dt}{t^{2n+1}} \right)^{1/2} \\ &\leq C \|b\|_* \|f\|_{L^{p,\kappa}(w)} \sum_{k=1}^{\infty} (k+1) \cdot w(2^{k+1}B)^{(\kappa-1)/p}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{w(B)^{\kappa/p}} \left( \int_B \text{IV}^p w(x) dx \right)^{1/p} &\leq C \|b\|_* \|f\|_{L^{p,\kappa}(w)} \sum_{k=1}^{\infty} (k+1) \cdot \frac{w(B)^{(1-\kappa)/p}}{w(2^{k+1}B)^{(1-\kappa)/p}} \\ &\leq C \|b\|_* \|f\|_{L^{p,\kappa}(w)} \sum_{k=1}^{\infty} \frac{k}{2^{kn\theta}} \\ &\leq C \|b\|_* \|f\|_{L^{p,\kappa}(w)}, \end{aligned} \quad (11)$$

where we have used the previous estimate (3) with  $w \in RH_r$  and  $\theta = (1 - \kappa)(r - 1)/pr$ . Summarizing the estimates (9) and (11) derived above, we thus obtain

$$\frac{1}{w(B)^{\kappa/p}} \left( \int_B \Pi^p w(x) dx \right)^{1/p} \leq C \|b\|_* \|f\|_{L^{p,\kappa}(w)}. \quad (12)$$

Combining the inequalities (4), (6) with the above inequality (12) and taking the supremum over all balls  $B \subseteq \mathbb{R}^n$ , we complete the proof of Theorem 1.2.  $\square$

## 4 Proofs of Theorems 1.3 and 1.4

In order to prove the main theorems of this section, we need to establish the following three lemmas.

**Lemma 4.1.** *Let  $0 < \alpha \leq 1$  and  $w \in A_p$  with  $p = 2$ . Then for any  $j \in \mathbb{Z}_+$ , we have*

$$\|\mathcal{S}_{\alpha,2^j}(f)\|_{L_w^2} \leq C \cdot 2^{jn} \|\mathcal{S}_\alpha(f)\|_{L_w^2}.$$

*Proof.* Since  $w \in A_2$ , then by Lemma 2.1, we get

$$w(B(y, 2^j t)) = w(2^j B(y, t)) \leq C \cdot 2^{2jn} w(B(y, t)) \quad j = 1, 2, \dots$$

Therefore

$$\begin{aligned} \|\mathcal{S}_{\alpha,2^j}(f)\|_{L_w^2}^2 &= \int_{\mathbb{R}^n} \left( \iint_{\mathbb{R}_+^{n+1}} \left( A_\alpha(f)(y, t) \right)^2 \chi_{|x-y| < 2^j t} \frac{dy dt}{t^{n+1}} \right) w(x) dx \\ &= \iint_{\mathbb{R}_+^{n+1}} \left( \int_{|x-y| < 2^j t} w(x) dx \right) \left( A_\alpha(f)(y, t) \right)^2 \frac{dy dt}{t^{n+1}} \\ &\leq C \cdot 2^{2jn} \iint_{\mathbb{R}_+^{n+1}} \left( \int_{|x-y| < t} w(x) dx \right) \left( A_\alpha(f)(y, t) \right)^2 \frac{dy dt}{t^{n+1}} \\ &= C \cdot 2^{2jn} \|\mathcal{S}_\alpha(f)\|_{L_w^2}^2. \end{aligned}$$

This finishes the proof of Lemma 4.1.  $\square$

**Lemma 4.2.** *Let  $0 < \alpha \leq 1$ ,  $2 < p < \infty$  and  $w \in A_p$ . Then for any  $j \in \mathbb{Z}_+$ , we have*

$$\|\mathcal{S}_{\alpha,2^j}(f)\|_{L_w^p} \leq C \cdot 2^{jnp/2} \|\mathcal{S}_\alpha(f)\|_{L_w^p}.$$

*Proof.* For any  $j \in \mathbb{Z}_+$ , it is easy to see that

$$\|\mathcal{S}_{\alpha,2^j}(f)\|_{L_w^p}^2 = \|\mathcal{S}_{\alpha,2^j}(f)^2\|_{L_w^{p/2}}.$$

Since  $p/2 > 1$ , then we have

$$\begin{aligned}
& \|\mathcal{S}_{\alpha,2^j}(f)^2\|_{L_w^{p/2}} \\
&= \sup_{\|g\|_{L_w^{(p/2)'}} \leq 1} \left| \int_{\mathbb{R}^n} \mathcal{S}_{\alpha,2^j}(f)(x)^2 g(x) w(x) dx \right| \\
&= \sup_{\|g\|_{L_w^{(p/2)'}} \leq 1} \left| \int_{\mathbb{R}^n} \left( \iint_{\mathbb{R}_+^{n+1}} \left( A_\alpha(f)(y,t) \right)^2 \chi_{|x-y| < 2^j t} \frac{dy dt}{t^{n+1}} \right) g(x) w(x) dx \right| \\
&= \sup_{\|g\|_{L_w^{(p/2)'}} \leq 1} \left| \iint_{\mathbb{R}_+^{n+1}} \left( \int_{|x-y| < 2^j t} g(x) w(x) dx \right) \left( A_\alpha(f)(y,t) \right)^2 \frac{dy dt}{t^{n+1}} \right|. \quad (13)
\end{aligned}$$

For  $w \in A_p$ , we denote the weighted maximal operator by  $M_w$ ; that is

$$M_w(f)(x) = \sup_{x \in B} \frac{1}{w(B)} \int_B |f(y)| w(y) dy,$$

where the supremum is taken over all balls  $B$  which contain  $x$ . Then, by Lemma 2.1, we can get

$$\begin{aligned}
\int_{|x-y| < 2^j t} g(x) w(x) dx &\leq C \cdot 2^{jnp} w(B(y,t)) \cdot \frac{1}{w(B(y,2^j t))} \int_{B(y,2^j t)} g(x) w(x) dx \\
&\leq C \cdot 2^{jnp} w(B(y,t)) \inf_{x \in B(y,t)} M_w(g)(x) \\
&\leq C \cdot 2^{jnp} \int_{|x-y| < t} M_w(g)(x) w(x) dx. \quad (14)
\end{aligned}$$

Substituting the above inequality (14) into (13) and using Hölder's inequality and the  $L_w^{(p/2)'}$  boundedness of  $M_w$ , we thus obtain

$$\begin{aligned}
\|\mathcal{S}_{\alpha,2^j}(f)^2\|_{L_w^{p/2}} &\leq C \cdot 2^{jnp} \sup_{\|g\|_{L_w^{(p/2)'}} \leq 1} \left| \int_{\mathbb{R}^n} \mathcal{S}_\alpha(f)(x)^2 M_w(g)(x) w(x) dx \right| \\
&\leq C \cdot 2^{jnp} \|\mathcal{S}_\alpha(f)^2\|_{L_w^{p/2}} \sup_{\|g\|_{L_w^{(p/2)'}} \leq 1} \|M_w(g)\|_{L_w^{(p/2)'}} \\
&\leq C \cdot 2^{jnp} \|\mathcal{S}_\alpha(f)^2\|_{L_w^{p/2}} \\
&= C \cdot 2^{jnp} \|\mathcal{S}_\alpha(f)\|_{L_w^p}^2.
\end{aligned}$$

This implies the desired result.  $\square$

**Lemma 4.3.** *Let  $0 < \alpha \leq 1$ ,  $1 < p < 2$  and  $w \in A_p$ . Then for any  $j \in \mathbb{Z}_+$ , we have*

$$\|\mathcal{S}_{\alpha,2^j}(f)\|_{L_w^p} \leq C \cdot 2^{jn} \|\mathcal{S}_\alpha(f)\|_{L_w^p}.$$

*Proof.* We will adopt the same method as in [18, page 315–316]. For any  $j \in \mathbb{Z}_+$ , set  $\Omega_\lambda = \{x \in \mathbb{R}^n : \mathcal{S}_\alpha(f)(x) > \lambda\}$  and  $\Omega_{\lambda,j} = \{x \in \mathbb{R}^n : \mathcal{S}_{\alpha,2^j}(f)(x) > \lambda\}$ . We also set

$$\Omega_\lambda^* = \left\{x \in \mathbb{R}^n : M_w(\chi_{\Omega_\lambda})(x) > \frac{1}{2^{(jn+1) \cdot [w]_{A_p}}}\right\}.$$

Observe that  $w(\Omega_{\lambda,j}) \leq w(\Omega_\lambda^*) + w(\Omega_{\lambda,j} \cap (\mathbb{R}^n \setminus \Omega_\lambda^*))$ . Thus

$$\begin{aligned} \|\mathcal{S}_{\alpha,2^j}(f)\|_{L_w^p}^p &= \int_0^\infty p\lambda^{p-1}w(\Omega_{\lambda,j}) d\lambda \\ &\leq \int_0^\infty p\lambda^{p-1}w(\Omega_\lambda^*) d\lambda + \int_0^\infty p\lambda^{p-1}w(\Omega_{\lambda,j} \cap (\mathbb{R}^n \setminus \Omega_\lambda^*)) d\lambda \\ &= \text{I} + \text{II}. \end{aligned}$$

The weighted weak type estimate of  $M_w$  yields

$$\text{I} \leq C \cdot 2^{jnp} \int_0^\infty p\lambda^{p-1}w(\Omega_\lambda) d\lambda = C \cdot 2^{jnp} \|\mathcal{S}_\alpha(f)\|_{L_w^p}^p. \quad (15)$$

To estimate II, we now claim that the following inequality holds.

$$\int_{\mathbb{R}^n \setminus \Omega_\lambda^*} \mathcal{S}_{\alpha,2^j}(f)(x)^2 w(x) dx \leq C \cdot 2^{jnp} \int_{\mathbb{R}^n \setminus \Omega_\lambda} \mathcal{S}_\alpha(f)(x)^2 w(x) dx. \quad (16)$$

We will take the above inequality temporarily for granted, then it follows from Chebyshev's inequality and (16) that

$$\begin{aligned} w(\Omega_{\lambda,j} \cap (\mathbb{R}^n \setminus \Omega_\lambda^*)) &\leq \lambda^{-2} \int_{\Omega_{\lambda,j} \cap (\mathbb{R}^n \setminus \Omega_\lambda^*)} \mathcal{S}_{\alpha,2^j}(f)(x)^2 w(x) dx \\ &\leq \lambda^{-2} \int_{\mathbb{R}^n \setminus \Omega_\lambda^*} \mathcal{S}_{\alpha,2^j}(f)(x)^2 w(x) dx \\ &\leq C \cdot 2^{jnp} \lambda^{-2} \int_{\mathbb{R}^n \setminus \Omega_\lambda} \mathcal{S}_\alpha(f)(x)^2 w(x) dx. \end{aligned}$$

Hence

$$\text{II} \leq C \cdot 2^{jnp} \int_0^\infty p\lambda^{p-1} \left( \lambda^{-2} \int_{\mathbb{R}^n \setminus \Omega_\lambda} \mathcal{S}_\alpha(f)(x)^2 w(x) dx \right) d\lambda.$$

Changing the order of integration yields

$$\begin{aligned} \text{II} &\leq C \cdot 2^{jnp} \int_{\mathbb{R}^n} \mathcal{S}_\alpha(f)(x)^2 \left( \int_{|\mathcal{S}_\alpha(f)(x)|}^\infty p\lambda^{p-3} d\lambda \right) w(x) dx \\ &\leq C \cdot 2^{jnp} \frac{p}{2-p} \cdot \|\mathcal{S}_\alpha(f)\|_{L_w^p}^p. \end{aligned} \quad (17)$$

Combining the above estimate (17) with (15) and taking  $p$ -th root on both sides, we complete the proof of Lemma 4.3. So it remains to prove the inequality (16).

Set  $\Gamma_{2^j}(\mathbb{R}^n \setminus \Omega_\lambda^*) = \bigcup_{x \in \mathbb{R}^n \setminus \Omega_\lambda^*} \Gamma_{2^j}(x)$  and  $\Gamma(\mathbb{R}^n \setminus \Omega_\lambda) = \bigcup_{x \in \mathbb{R}^n \setminus \Omega_\lambda} \Gamma(x)$ . For each given  $(y, t) \in \Gamma_{2^j}(\mathbb{R}^n \setminus \Omega_\lambda^*)$ , by Lemma 2.1, we thus have

$$w(B(y, 2^j t) \cap (\mathbb{R}^n \setminus \Omega_\lambda^*)) \leq C \cdot 2^{jnp} w(B(y, t)).$$

It is not difficult to check that  $w(B(y, t) \cap \Omega_\lambda) \leq \frac{w(B(y, t))}{2}$  and  $\Gamma_{2^j}(\mathbb{R}^n \setminus \Omega_\lambda^*) \subseteq \Gamma(\mathbb{R}^n \setminus \Omega_\lambda)$ . In fact, for any  $(y, t) \in \Gamma_{2^j}(\mathbb{R}^n \setminus \Omega_\lambda^*)$ , there exists a point  $x \in \mathbb{R}^n \setminus \Omega_\lambda^*$  such that  $(y, t) \in \Gamma_{2^j}(x)$ . Then we can deduce

$$\begin{aligned} w(B(y, t) \cap \Omega_\lambda) &\leq w(B(y, 2^j t) \cap \Omega_\lambda) \\ &= \int_{B(y, 2^j t)} \chi_{\Omega_\lambda}(z) w(z) dz \\ &\leq [w]_{A_p} \cdot 2^{jnp} w(B(y, t)) \cdot \frac{1}{w(B(y, 2^j t))} \int_{B(y, 2^j t)} \chi_{\Omega_\lambda}(z) w(z) dz. \end{aligned}$$

Note that  $x \in B(y, 2^j t) \cap (\mathbb{R}^n \setminus \Omega_\lambda^*)$ . So we have

$$w(B(y, t) \cap \Omega_\lambda) \leq [w]_{A_p} \cdot 2^{jnp} w(B(y, t)) M_w(\chi_{\Omega_\lambda})(x) \leq \frac{w(B(y, t))}{2}.$$

Hence

$$\begin{aligned} w(B(y, t)) &= w(B(y, t) \cap \Omega_\lambda) + w(B(y, t) \cap (\mathbb{R}^n \setminus \Omega_\lambda)) \\ &\leq \frac{w(B(y, t))}{2} + w(B(y, t) \cap (\mathbb{R}^n \setminus \Omega_\lambda)), \end{aligned}$$

which is equivalent to

$$w(B(y, t)) \leq 2 \cdot w(B(y, t) \cap (\mathbb{R}^n \setminus \Omega_\lambda)).$$

The above inequality implies in particular that there is a point  $z \in B(y, t) \cap (\mathbb{R}^n \setminus \Omega_\lambda) \neq \emptyset$ . In this case, we have  $(y, t) \in \Gamma(z)$  with  $z \in \mathbb{R}^n \setminus \Omega_\lambda$ , which yields  $\Gamma_{2^j}(\mathbb{R}^n \setminus \Omega_\lambda^*) \subseteq \Gamma(\mathbb{R}^n \setminus \Omega_\lambda)$ . Thus we obtain

$$w(B(y, 2^j t) \cap (\mathbb{R}^n \setminus \Omega_\lambda^*)) \leq C \cdot 2^{jnp} w(B(y, t) \cap (\mathbb{R}^n \setminus \Omega_\lambda)).$$

Therefore

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus \Omega_\lambda^*} \mathcal{S}_{\alpha, 2^j}(f)(x)^2 w(x) dx \\ &= \int_{\mathbb{R}^n \setminus \Omega_\lambda^*} \left( \iint_{\Gamma_{2^j}(x)} \left( A_\alpha(f)(y, t) \right)^2 \frac{dy dt}{t^{n+1}} \right) w(x) dx \\ &\leq \iint_{\Gamma_{2^j}(\mathbb{R}^n \setminus \Omega_\lambda^*)} \left( \int_{B(y, 2^j t) \cap (\mathbb{R}^n \setminus \Omega_\lambda^*)} w(x) dx \right) \left( A_\alpha(f)(y, t) \right)^2 \frac{dy dt}{t^{n+1}} \\ &\leq C \cdot 2^{jnp} \iint_{\Gamma(\mathbb{R}^n \setminus \Omega_\lambda)} \left( \int_{B(y, t) \cap (\mathbb{R}^n \setminus \Omega_\lambda)} w(x) dx \right) \left( A_\alpha(f)(y, t) \right)^2 \frac{dy dt}{t^{n+1}} \\ &\leq C \cdot 2^{jnp} \int_{\mathbb{R}^n \setminus \Omega_\lambda} \mathcal{S}_\alpha(f)(x)^2 w(x) dx, \end{aligned}$$

which is just our desired conclusion.  $\square$

We are now in a position to give the proof of Theorem 1.3.

*Proof of Theorem 1.3.* From the definition of  $g_{\lambda,\alpha}^*$ , we readily see that

$$\begin{aligned}
g_{\lambda,\alpha}^*(f)(x)^2 &= \iint_{\mathbb{R}_+^{n+1}} \left( \frac{t}{t+|x-y|} \right)^{\lambda n} \left( A_\alpha(f)(y,t) \right)^2 \frac{dydt}{t^{n+1}} \\
&= \int_0^\infty \int_{|x-y|<t} \left( \frac{t}{t+|x-y|} \right)^{\lambda n} \left( A_\alpha(f)(y,t) \right)^2 \frac{dydt}{t^{n+1}} \\
&\quad + \sum_{j=1}^\infty \int_0^\infty \int_{2^{j-1}t \leq |x-y| < 2^j t} \left( \frac{t}{t+|x-y|} \right)^{\lambda n} \left( A_\alpha(f)(y,t) \right)^2 \frac{dydt}{t^{n+1}} \\
&\leq C \left[ \mathcal{S}_\alpha(f)(x)^2 + \sum_{j=1}^\infty 2^{-j\lambda n} \mathcal{S}_{\alpha,2^j}(f)(x)^2 \right].
\end{aligned}$$

For any given ball  $B = B(x_0, r_B) \subseteq \mathbb{R}^n$ , then from the above inequality, it follows that

$$\begin{aligned}
&\frac{1}{w(B)^{\kappa/p}} \left( \int_B |g_{\lambda,\alpha}^*(f)(x)|^p w(x) dx \right)^{1/p} \\
&\leq \frac{1}{w(B)^{\kappa/p}} \left( \int_B |\mathcal{S}_\alpha(f)(x)|^p w(x) dx \right)^{1/p} + \sum_{j=1}^\infty 2^{-j\lambda n/2} \cdot \frac{1}{w(B)^{\kappa/p}} \left( \int_B |\mathcal{S}_{\alpha,2^j}(f)(x)|^p w(x) dx \right)^{1/p} \\
&= I_0 + \sum_{j=1}^\infty 2^{-j\lambda n/2} I_j.
\end{aligned}$$

By Theorem 1.1, we know that  $I_0 \leq C \|f\|_{L^{p,\kappa}(w)}$ . Below we shall give the estimates of  $I_j$  for  $j = 1, 2, \dots$ . As before, we set  $f = f_1 + f_2$ ,  $f_1 = f\chi_{2B}$  and write

$$\begin{aligned}
&\frac{1}{w(B)^{\kappa/p}} \left( \int_B |\mathcal{S}_{\alpha,2^j}(f)(x)|^p w(x) dx \right)^{1/p} \\
&\leq \frac{1}{w(B)^{\kappa/p}} \left( \int_B |\mathcal{S}_{\alpha,2^j}(f_1)(x)|^p w(x) dx \right)^{1/p} + \frac{1}{w(B)^{\kappa/p}} \left( \int_B |\mathcal{S}_{\alpha,2^j}(f_2)(x)|^p w(x) dx \right)^{1/p} \\
&= I_j^{(1)} + I_j^{(2)}.
\end{aligned}$$

Applying Lemmas 4.1–4.3, Theorem A and Lemma 2.1, we obtain

$$\begin{aligned}
I_j^{(1)} &\leq \frac{1}{w(B)^{\kappa/p}} \|\mathcal{S}_{\alpha,2^j}(f_1)\|_{L_w^p} \\
&\leq C \left( 2^{jn} + 2^{jnp/2} \right) \frac{1}{w(B)^{\kappa/p}} \cdot \|f_1\|_{L_w^p} \\
&\leq C \|f\|_{L^{p,\kappa}(w)} \left( 2^{jn} + 2^{jnp/2} \right) \cdot \frac{w(2B)^{\kappa/p}}{w(B)^{\kappa/p}} \\
&\leq C \|f\|_{L^{p,\kappa}(w)} \left( 2^{jn} + 2^{jnp/2} \right).
\end{aligned}$$



We now turn to estimate the term  $I_j^{(2)}$ . For any  $x \in B$ ,  $(y, t) \in \Gamma_{2^j}(x)$  and  $z \in (2^{k+1}B \setminus 2^k B) \cap B(y, t)$ , then by a direct calculation, we can easily deduce

$$t + 2^j t \geq |x - y| + |y - z| \geq |x - z| \geq |z - x_0| - |x - x_0| \geq 2^{k-1} r_B.$$

Thus, it follows from the previous estimates (1) and (2) that

$$\begin{aligned} |\mathcal{S}_{\alpha, 2^j}(f_2)(x)| &= \left( \iint_{\Gamma_{2^j}(x)} \sup_{\varphi \in \mathcal{C}_\alpha} |f_2 * \varphi_t(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} \\ &\leq C \left( \int_{2^{(k-2-j)} r_B}^\infty \int_{|x-y| < 2^j t} \left| t^{-n} \sum_{k=1}^\infty \int_{2^{k+1}B \setminus 2^k B} |f(z)| dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} \\ &\leq C \left( \sum_{k=1}^\infty \int_{2^{k+1}B \setminus 2^k B} |f(z)| dz \right) \left( \int_{2^{(k-2-j)} r_B}^\infty 2^{jn} \frac{dt}{t^{2n+1}} \right)^{1/2} \\ &\leq C \cdot 2^{3jn/2} \sum_{k=1}^\infty \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B \setminus 2^k B} |f(z)| dz \\ &\leq C \|f\|_{L^{p, \kappa}(w)} \cdot 2^{3jn/2} \sum_{k=1}^\infty w(2^{k+1}B)^{(\kappa-1)/p}. \end{aligned}$$

Furthermore, by using (3) again, we get

$$\begin{aligned} I_j^{(2)} &\leq C \|f\|_{L^{p, \kappa}(w)} \cdot 2^{3jn/2} \sum_{k=1}^\infty \frac{w(B)^{(1-\kappa)/p}}{w(2^{k+1}B)^{(1-\kappa)/p}} \\ &\leq C \|f\|_{L^{p, \kappa}(w)} \cdot 2^{3jn/2}. \end{aligned}$$

Therefore

$$\begin{aligned} &\frac{1}{w(B)^{\kappa/p}} \left( \int_B |g_{\lambda, \alpha}^*(f)(x)|^p w(x) dx \right)^{1/p} \\ &\leq C \|f\|_{L^{p, \kappa}(w)} \left( 1 + \sum_{j=1}^\infty 2^{-j\lambda n/2} 2^{3jn/2} + \sum_{j=1}^\infty 2^{-j\lambda n/2} 2^{jnp/2} \right) \\ &\leq C \|f\|_{L^{p, \kappa}(w)}, \end{aligned}$$

where the last two series are both convergent under our assumption  $\lambda > \max\{p, 3\}$ . Hence, by taking the supremum over all balls  $B \subseteq \mathbb{R}^n$ , we conclude the proof of Theorem 1.3.  $\square$

Finally, we remark that by using the arguments as in the proof of Theorems 1.2 and 1.3, we can also show the conclusion of Theorem 1.4. The details are omitted here.

## References

- [1] D. R. Adams, A note on Riesz potentials, *Duke Math. J.*, **42**(1975), 765–778.
- [2] J. Alvarez, R. J. Bagby, D. S. Kurtz and C. Pérez, Weighted estimates for commutators of linear operators, *Studia Math.*, **104**(1993), 195–209.
- [3] F. Chiarenza and M. Frasca, Morrey spaces and Hardy-Littlewood maximal function, *Rend. Math. Appl.*, **7**(1987), 273–279.
- [4] Y. Ding, S. Z. Lu and K. Yabuta, On commutators of Marcinkiewicz integrals with rough kernel, *J. Math. Anal. Appl.*, **275**(2002), 60–68.
- [5] J. Duoandikoetxea, *Fourier Analysis*, American Mathematical Society, Providence, Rhode Island, 2000.
- [6] D. S. Fan, S. Z. Lu and D. C. Yang, Regularity in Morrey spaces of strong solutions to nondivergence elliptic equations with VMO coefficients, *Georgian Math. J.*, **5**(1998), 425–440.
- [7] G. Di Fazio and M. A. Ragusa, Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients, *J. Funct. Anal.*, **112**(1993), 241–256.
- [8] G. Di Fazio, D. K. Palagachev and M. A. Ragusa, Global Morrey regularity of strong solutions to the Dirichlet problem for elliptic equations with discontinuous coefficients, *J. Funct. Anal.*, **166**(1999), 179–196.
- [9] J. Garcia-Cuerva and J. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, Amsterdam, 1985.
- [10] R. F. Gundy and R. L. Wheeden, Weighted integral inequalities for non-tangential maximal function, *Lusin area integral*, and *Walsh-Paley series*, *Studia Math.*, **49**(1974), 107–124.
- [11] J. Z. Huang and Y. Liu, Some characterizations of weighted Hardy spaces, *J. Math. Anal. Appl.*, **363**(2010), 121–127.
- [12] F. John and L. Nirenberg, On functions of bounded mean oscillation, *Comm. Pure Appl. Math.*, **14**(1961), 415–426.
- [13] Y. Komori and S. Shirai, Weighted Morrey spaces and a singular integral operator, *Math. Nachr.*, **282**(2009), 219–231.
- [14] A. K. Lerner, Sharp weighted norm inequalities for Littlewood-Paley operators and singular integrals, *Adv. Math.*, **226**(2011), 3912–3926.
- [15] C. B. Morrey, On the solutions of quasi-linear elliptic partial differential equations, *Trans. Amer. Math. Soc.*, **43**(1938), 126–166.
- [16] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, *Trans. Amer. Math. Soc.*, **165**(1972), 207–226.

- [17] J. Peetre, On the theory of  $\mathcal{L}_{p,\lambda}$  spaces, *J. Funct. Anal.*, **4**(1969), 71–87.
- [18] A. Torchinsky, *Real-Variable Methods in Harmonic Analysis*, Academic Press, New York, 1986.
- [19] H. Wang, The boundedness of some operators with rough kernel on the weighted Morrey spaces, *Acta Math. Sinica (Chin. Ser)*, **55**(2012), 589–600.
- [20] H. Wang, Boundedness of fractional integral operators with rough kernels on weighted Morrey spaces, preprint, 2012.
- [21] H. Wang and H. P. Liu, Some estimates for Bochner-Riesz operators on the weighted Morrey spaces, *Acta Math. Sinica (Chin. Ser)*, **55**(2012), 551–560.
- [22] H. Wang and H. P. Liu, The intrinsic square function characterizations of weighted Hardy spaces, *Illinois J. Math.*, to appear.
- [23] H. Wang and H. P. Liu, Weak type estimates of intrinsic square functions on the weighted Hardy spaces, *Arch. Math.*, **97**(2011), 49–59.
- [24] M. Wilson, The intrinsic square function, *Rev. Mat. Iberoamericana*, **23**(2007), 771–791.
- [25] M. Wilson, *Weighted Littlewood-Paley Theory and Exponential-Square Integrability*, *Lecture Notes in Math*, Vol 1924, Springer-Verlag, 2007.